Safe Probabilistic Invariance Verification for Stochastic Discrete-time Dynamical Systems

Colloquium on Design and Verification of Cyber-Physical Systems: From Theory to Applications
(Prof. Martin Fränzle's 60th birthday)

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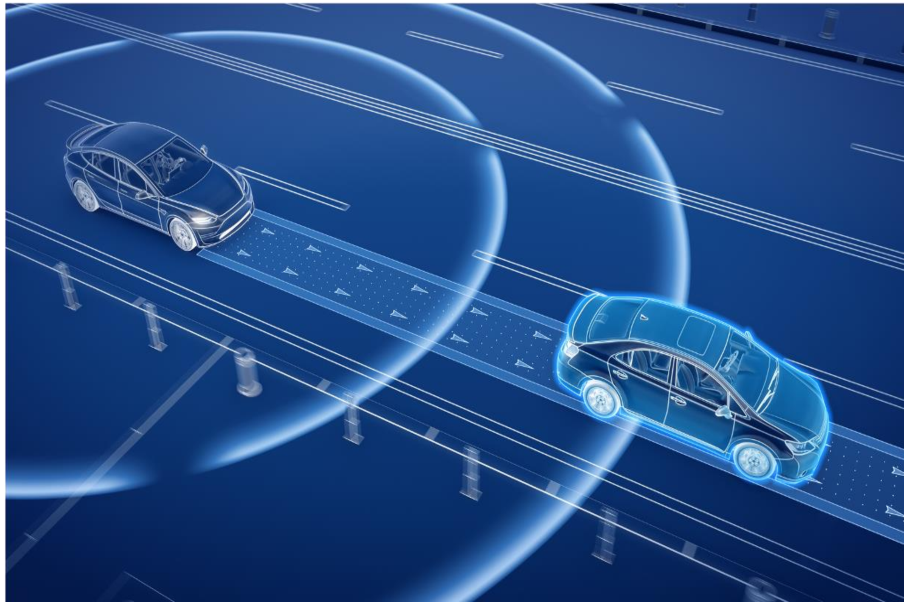
Background



Safety Is Paramount!

A safety property asserts that some "bad thing" does not happen during execution [Leslie Lamport, 1977]

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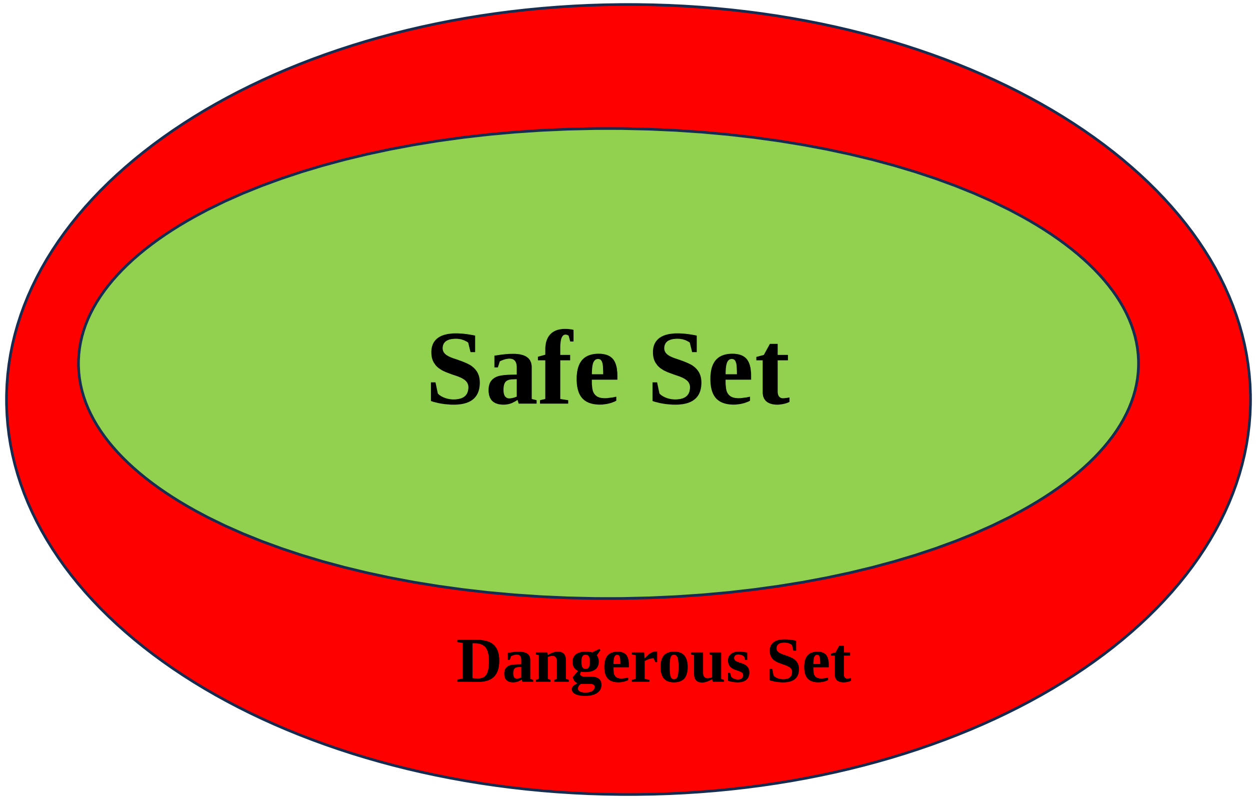

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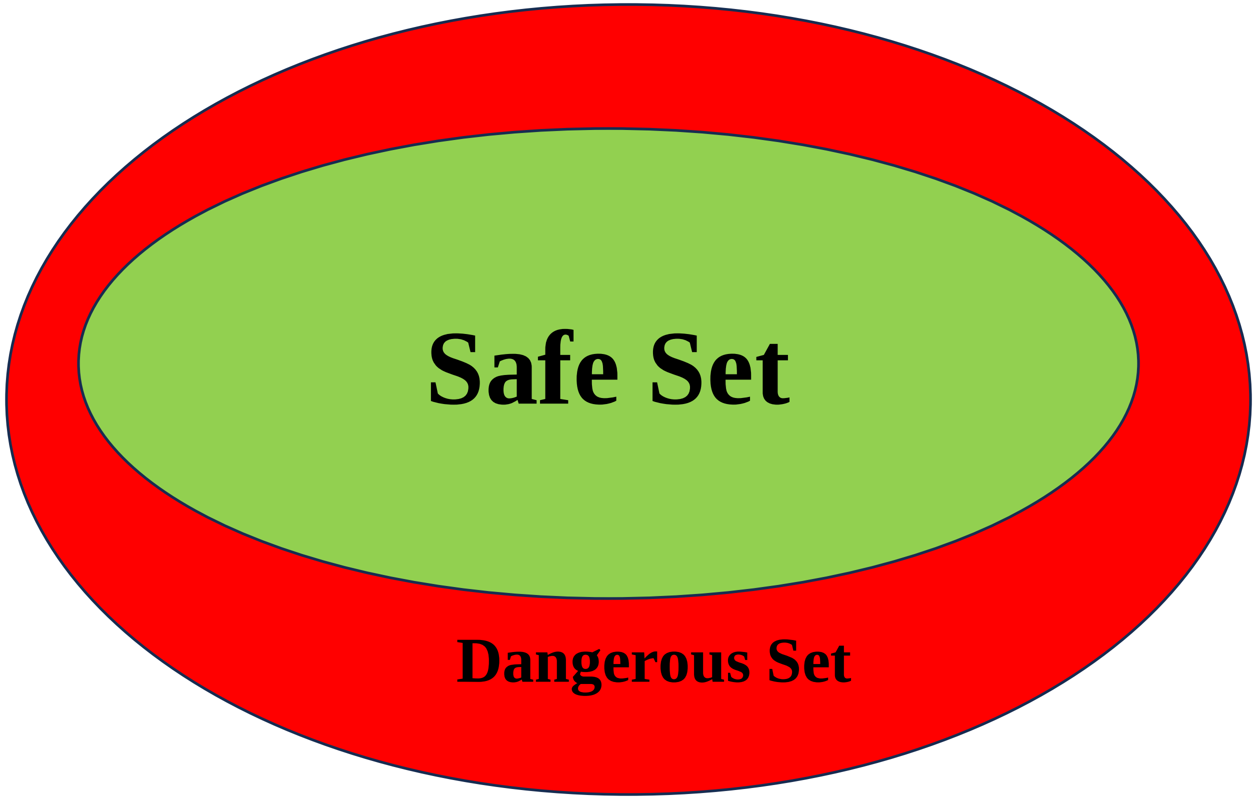
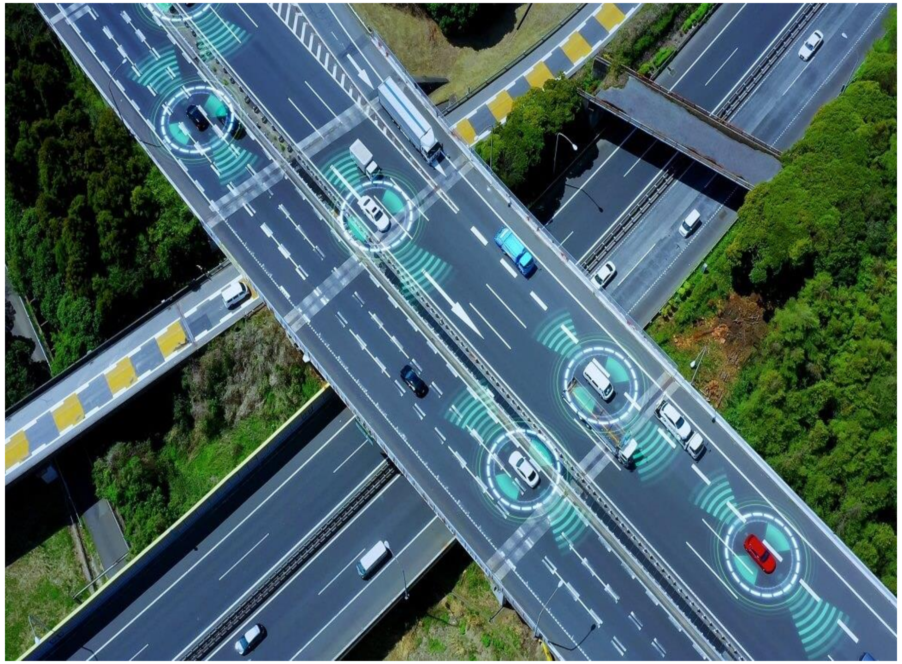


Dynamical systems:

- Discrete-time Systems: $x(k+1) = f\big(x(k)\big)$

The real system is highly complex: the complete information of such a system can indeed be challenging to acquire, especially in open environments.



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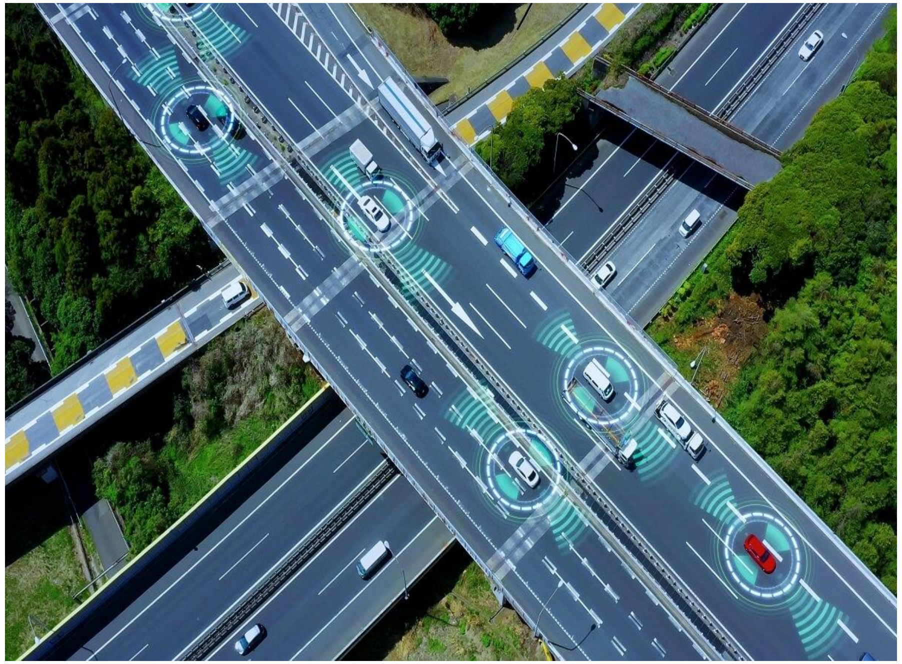

How do we guarantee safety of these systems?

Safe Robust Invariance Verification

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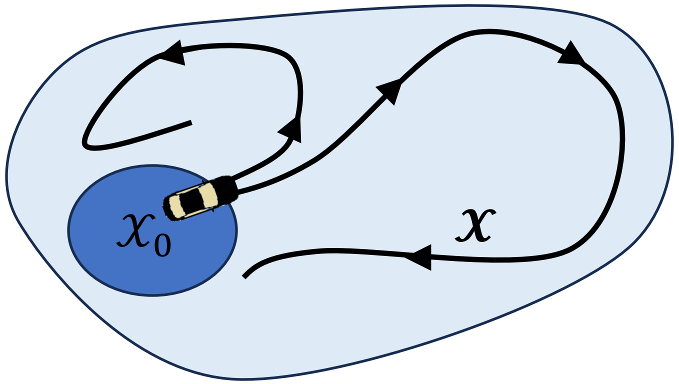
Safe robust invariance verification:

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, to verify that the system starting from $\mathcal{X}_0$ will **remain inside the safe set $\mathcal{X}$ for all time, regardless of disturbances**



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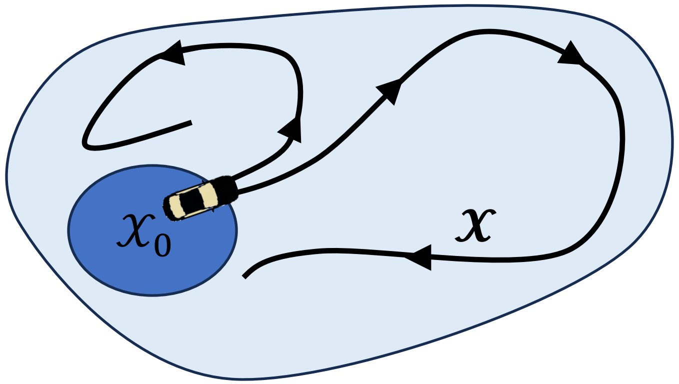
Given $\lambda > 0$, **finding Barrier Functions $B(x)$:**

$$\begin{cases} B(x) \geq 0, & \forall x \in \mathcal{X}_0, \\ B\big(f(x,d)\big) \geq \lambda B(x), & \forall x \in \mathcal{X}, \forall d \in \mathcal{D}, \\ B(x) < 0, & \forall x \in \mathbb{R}^n \setminus \mathcal{X}. \end{cases}$$



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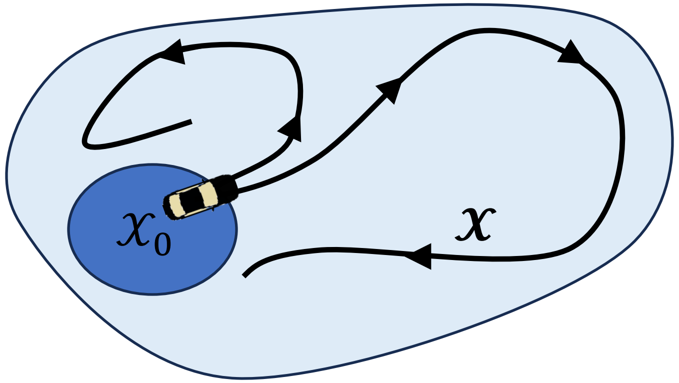

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Too Conservative!

Safe Probabilistic Invariance Verification

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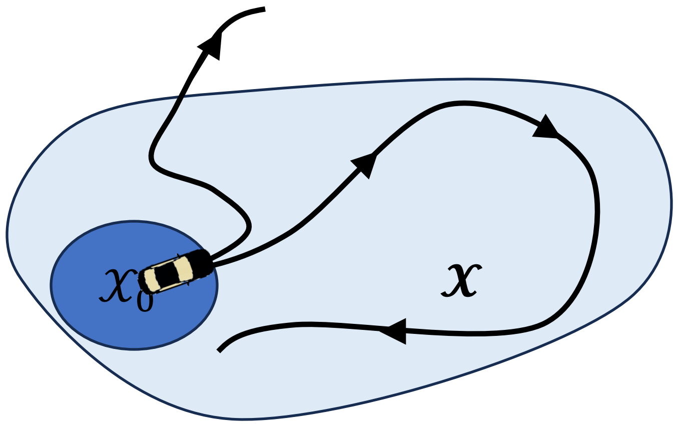
This approach reduces conservatism by allowing probabilistic violations



Problem Formulation

Stochastic Discrete-time Systems

$$x(k+1) = f\big(x(k), d(k)\big), \quad x(0) = x_0$$

- $d(k) \in \mathcal{D}$ is the stochastic perturbation input.
- $d(0), d(1), \ldots,$ are independent and identically distributed (i.i.d) on a probability space $(\boldsymbol{\mathcal{D}}, \mathcal{F}, \mathbb{P})$, with support $\boldsymbol{\mathcal{D}}$: for any measurable set $B \subseteq \boldsymbol{\mathcal{D}}$, $\mathrm{Prob}(d(l) \in B) = \mathbb{P}(B), \forall l \in \mathbb{N}$. The expectation is denoted by $\mathbb{E}[\cdot]$.

A disturbance signal $\boldsymbol{\pi}$ is a function $d(\cdot) : \mathbb{N} \to \mathcal{D}$: a sample path of a stochastic process defined on the canonical sample space $\boldsymbol{\Omega}^{\infty} = \boldsymbol{\mathcal{D}} \times \boldsymbol{\mathcal{D}} \times \cdots$ with the probability measure $\mathbb{P}^{\infty} = \mathbb{P} \times \mathbb{P} \times \cdots$.

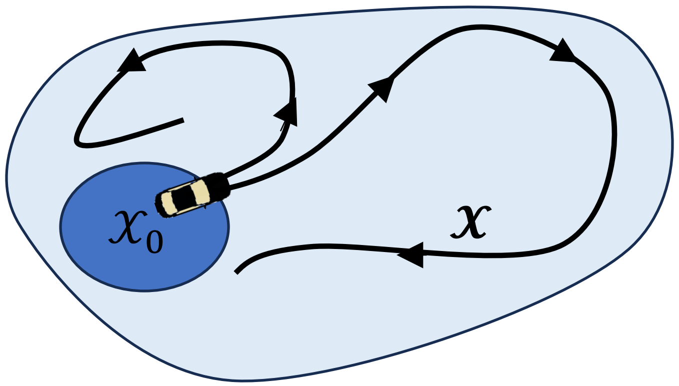
Trajectory $\boldsymbol{\phi}_{\boldsymbol{\pi}}^{x_0}(\cdot): \mathbb{N} \to \mathbb{R}^n$**:** $\boldsymbol{\phi_{\pi}^{x_0}(k+1) = f\left(\phi_{\pi}^{x_0}(k), \pi(k)\right), \phi_{\pi}^{x_0}(0) = x_0}$

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0 \subseteq \mathcal{X}$,

the safe probabilistic invariance verification is to compute lower and upper bounds, denoted by $\epsilon_1 \in [0,1]$ and $\epsilon_2 \in [0,1]$ respectively, for the safety probability that the system, starting from any state in $\mathcal{X}_0$, will remain inside the safe set $\mathcal{X}$ for all time, i.e.,

to compute ϵ_1 and ϵ_2 such that

$$\epsilon_1 \leq \mathbb{P}^{\infty}\Big(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x_0}}(\boldsymbol{k}) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0 \Big) \leq \epsilon_2.$$



Doob's Supermartingale Inequality Based Method

Doob's Supermartingale Inequality [J. Ville, 1939]

Let $(\Omega_1, \mathcal{F}, \mathbb{P}_1)$ be the probability space and $\{B_i\}_{i\in\mathbb{N}}$ be a non-negative supermartingale, then for $b > 0$,

$$\mathbb{P}_1\left(\sup_{i\in\mathbb{N}} B_i \geq b \mid B_0\right) \leq \frac{B_0}{b}.$$

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Under the assumption that $\Omega \subset \mathbb{R}^n$ is a robust invariant set, i.e., $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{d}): \boldsymbol{\Omega}\times\boldsymbol{\mathcal{D}} \to \boldsymbol{\Omega}$, and $\mathcal{X}_0 \subseteq \Omega$, if there exists $v(x): \Omega \to \mathbb{R}$ such that

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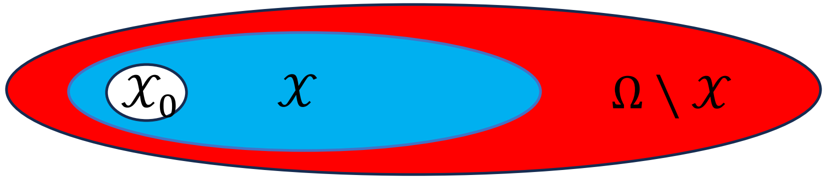
where $\boldsymbol{\Omega} \setminus \boldsymbol{\mathcal{X}}$ is a set of unsafe states, then

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x_0}}(\boldsymbol{k}) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \geq \epsilon_1.$$



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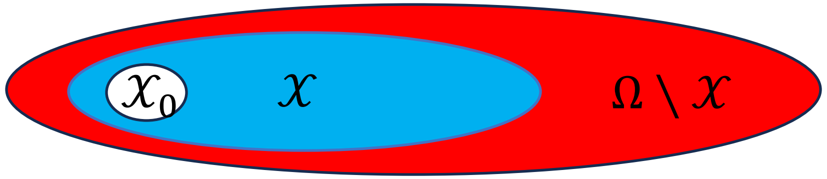

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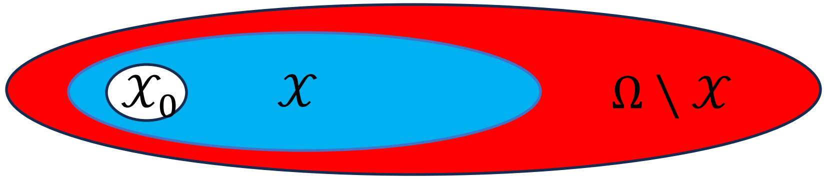

- $\boldsymbol{\Omega} \neq \mathbb{R}^{n}$: **challening to compute (if it exists)**
- $\boldsymbol{\Omega} = \mathbb{R}^{n}$: **producing conservative lower bounds**

Running Example

A computer-based model, which is modified from the reversed-time Van der Pol oscillator based on Euler's method with the time step 0.01:

$$\begin{cases} x(l+1) = x(l) - 0.02y(l), \\ y(l+1) = y(l) + 0.01\Big(\big(0.8 + d(l)\big)x(l) + 10\big(x^2(l) - 0.21\big)y(l)\Big). \end{cases}$$

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❖ Monto-Carlo method: $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x_0}}(\boldsymbol{k}) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\big) \approx \mathbf{1}$

❖ Method in [M. Anand, et. al., HSCC 2022]($\boldsymbol{\Omega} = \mathbb{R}^{\mathbf{2}}$)+ semi-definite programming tool:

$$\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x_0}}(\boldsymbol{k}) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\big) \geq \mathbf{2.1368e-07}$$

Doob's Supermartingale Inequality Based Method

An auxiliary system

$$x(k+1) = \tilde{f}\big(x(k), d(k)\big)$$

with

$$\tilde{f}(x,d) = f(x,d) \cdot 1_{\mathcal{X}}(x) + x \cdot 1_{\widetilde{\mathcal{X}} \setminus \mathcal{X}}(x)$$

$\widetilde{\mathcal{X}}$ is a set containing the union of the set $\mathcal{X}$ and all reachable states starting from $\mathcal{X}$ within one step:

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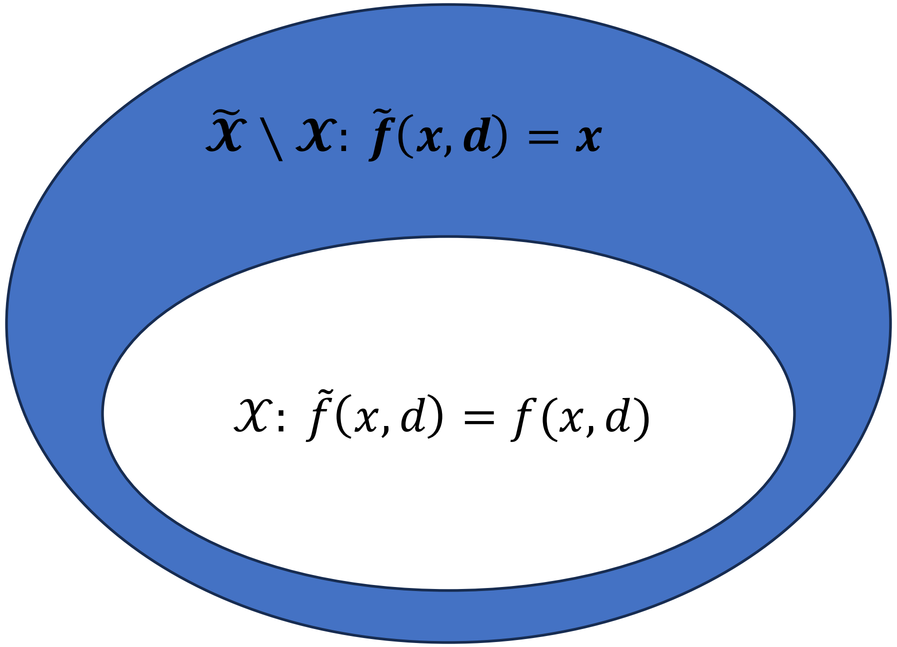

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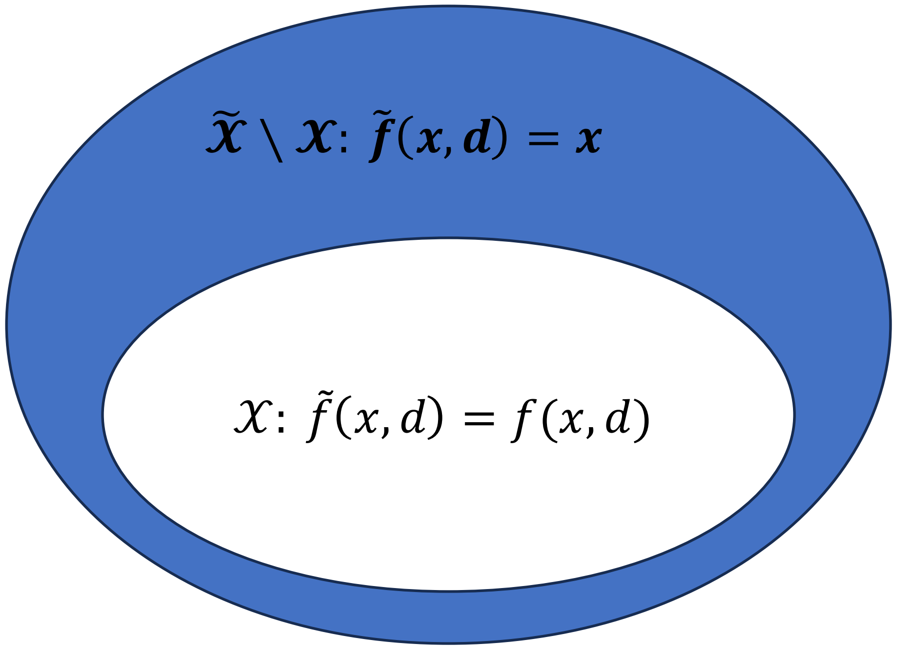

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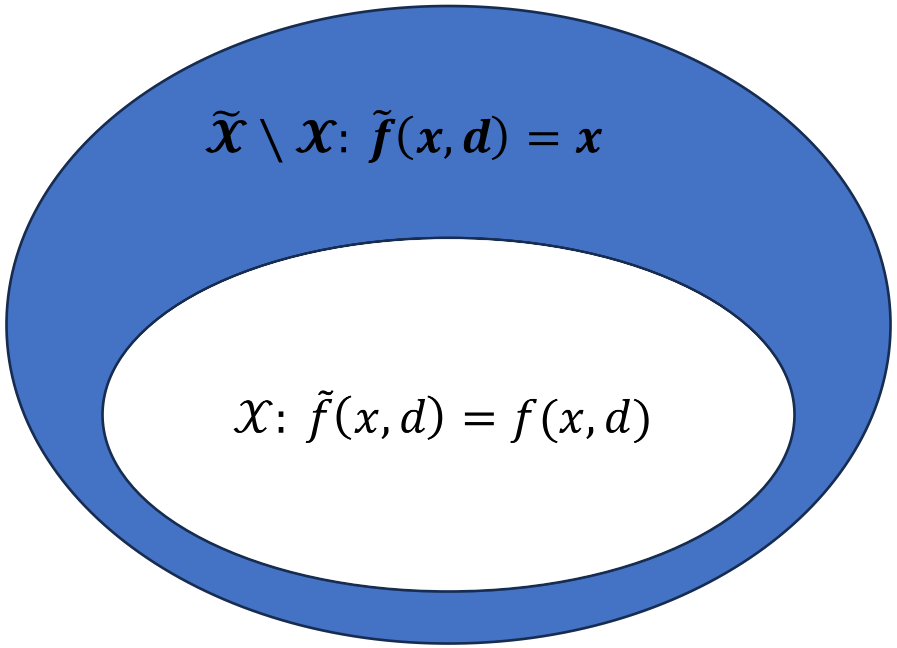

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Proposition 1 $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\ \phi_{\pi}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\big) = \mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\ \tilde{\phi}_{\pi}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\big)$

中国科学院软件研究所
Institute of Software Chinese Academy of Sciences

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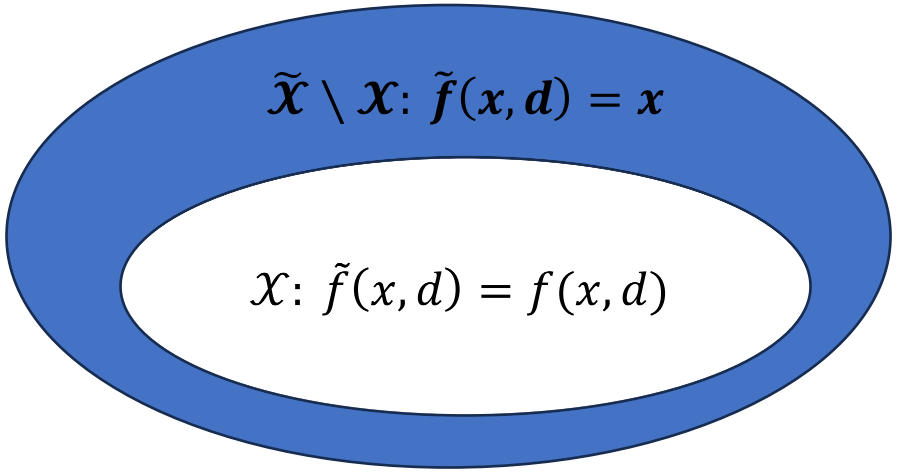

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$$\tilde{f}(x,d) = f(x,d) \cdot 1_{\mathcal{X}}(x) + x \cdot 1_{\widetilde{\mathcal{X}} \setminus \mathcal{X}}(x)$$

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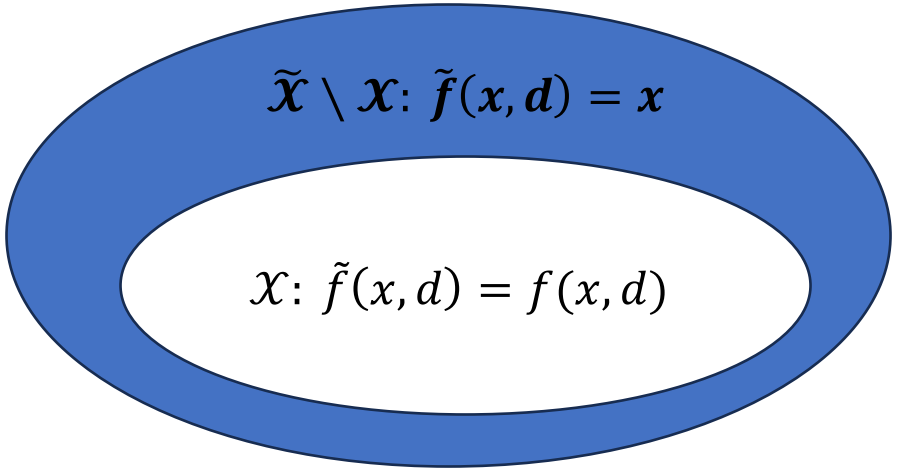

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- $\mathcal{X} = \{(x, y) \mid x^2 + y^2 - 1 \leq 0\}$
- $\mathcal{X}_0 = \{(x, y) \mid x^2 + y^2 - 0.01 < 0\}$

❖ Monto-Carlo method: $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x_0}}(\boldsymbol{k}) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\big) \approx \mathbf{1}$

❖ Method in [M. Anand, et. al., HSCC 2022]($\boldsymbol{\Omega} = \mathbb{R}^{\mathbf{2}}$)+ semi-definite programming tool:

$$\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x_0}}(\boldsymbol{k}) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\big) \geq \mathbf{2.1368}\boldsymbol{e} - \mathbf{07}$$

❖ **Our method** ($\widetilde{\mathcal{X}} = \{(x, y) \mid \boldsymbol{x^2} + \boldsymbol{y^2} - \mathbf{2} \leq \mathbf{0}\}$) +semi-definite programming tool:

$$\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x_0}}(\boldsymbol{k}) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\big) \geq \mathbf{0.9465}$$

Doob's Supermartingale Inequality Based Method

In [M. Anand, et. al., HSCC 2022],
Under the assumption that $\Omega \subset \mathbb{R}^n$ is a robust invariant set, i.e., $f(x,d): \Omega \times \mathcal{D} \to \Omega$, and $\mathcal{X} \subseteq \Omega$, if there exists $v(x): \Omega \to \mathbb{R}$ such that

$$\begin{cases} v(\boldsymbol{x}) \le \epsilon_2, & \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(x) \ge 0, & \forall \boldsymbol{x} \in \boldsymbol{\Omega}, \\ \mathbb{E}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{d}))] - v(\boldsymbol{x}) \le -\delta, & \forall \boldsymbol{x} \in \overline{\boldsymbol{\Omega} \setminus (\boldsymbol{\Omega} \setminus \mathcal{X})}, \\ v(\boldsymbol{x}) \ge 1, & \forall \boldsymbol{x} \in \partial\boldsymbol{\Omega} \setminus \partial(\boldsymbol{\Omega} \setminus \mathcal{X}), \end{cases}$$

then $\mathbb{P}^\infty\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0 \right) \le \epsilon_2$.

Theorem 2 Let $\mathcal{X}$ be a closed set. If there exists $v(x): \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ such that

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then
$\mathbb{P}^\infty\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0 \right) \le \epsilon_2$.

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If $v(x)$ is bounded over $\widetilde{\mathcal{X}}$, it provides strong guarantees of leaving the safe set $\mathcal{X}$ almost surely, i.e., $\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \phi_{\pi}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0 \right) = 0$.

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Equation Relaxation Based Method

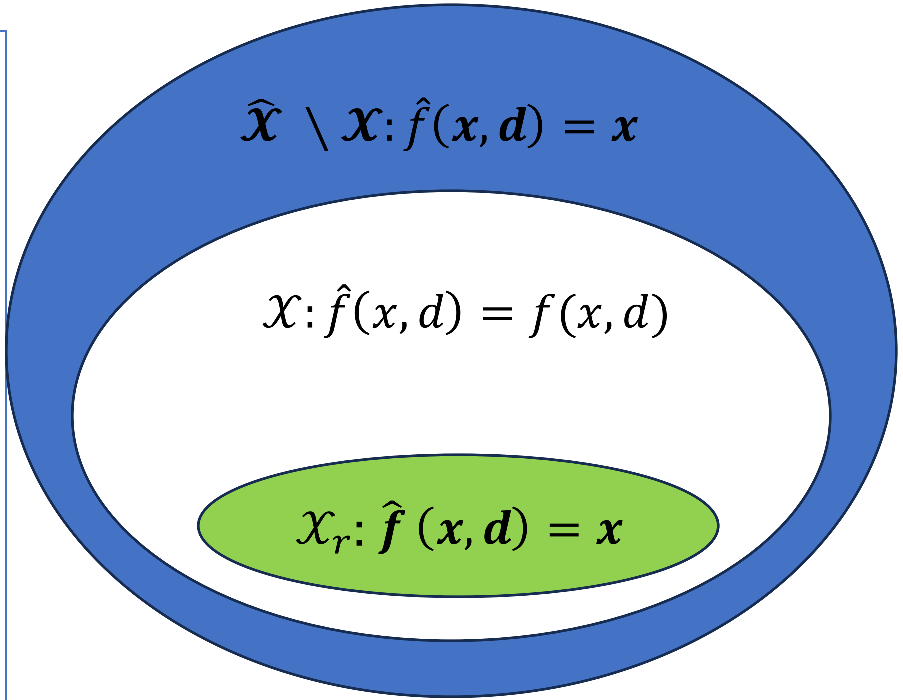
In [B. Xue, et. al., ACC 2021],
Given a safe set $\mathcal{X}$, a target set $\mathcal{X}_r$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_r, \mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x)$: $\widehat{\mathcal{X}} \to \mathbb{R}$ and $w(x)$: $\widehat{\mathcal{X}} \to \mathbb{R}$ such that

$$\begin{cases} v(x) = \mathbb{E}\left[v\left(\hat{f}\,(x,d)\right)\right], \forall x \in \widehat{\mathcal{X}}, \\ v(x) = 1_{\mathcal{X}_r}(x) + \mathbb{E}\left[w\left(\hat{f}\,(x,d)\right)\right] - w(x), \forall x \in \widehat{\mathcal{X}}. \end{cases}$$

Then,

$$\mathbb{P}^{\infty}\left(\exists k \in \mathbb{N}.\ \phi_{\pi}^{x_0}(k) \in \mathcal{X}_r \wedge \forall l \in [0,k].\ \phi_{\pi}^{x_0}(l) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) = v(x_0),$$

where $\hat{f}\,(x,d) = f(x,d) \cdot 1_{\mathcal{X}}(x) + x \cdot 1_{\widetilde{\mathcal{X}} \setminus \mathcal{X}}(x) + x \cdot 1_{\mathcal{X}_r}(x)$



B. Xue, R. Li, N. Zhan, and M. Fränzle. Reach-avoid analysis for stochastic discrete-time systems. In 2021 American Control Conference (ACC), pages 4879–4885. IEEE, 2021

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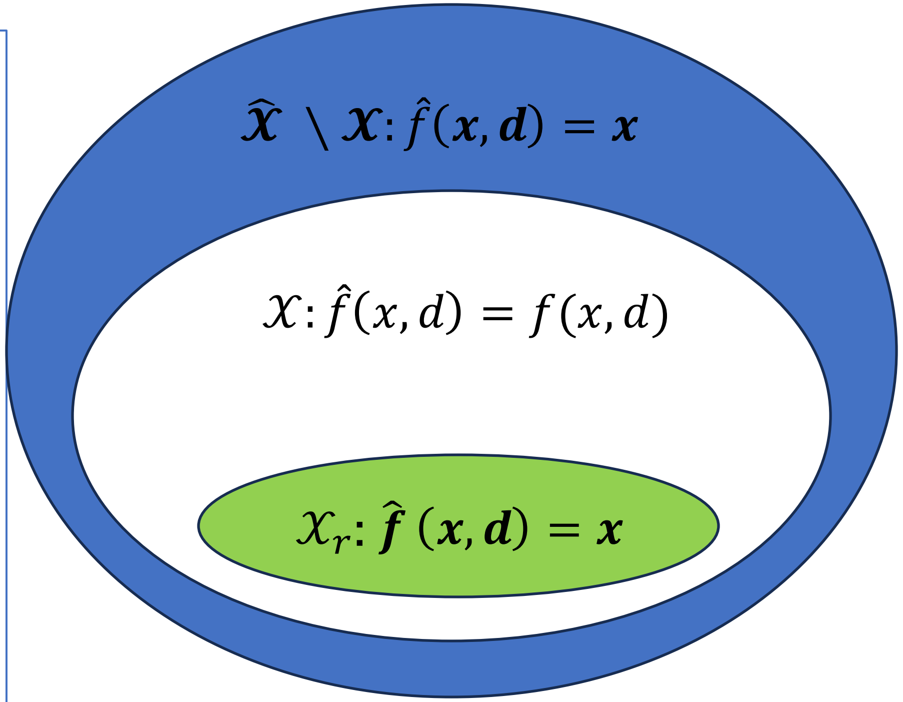
Then,
$\mathbb{P}^{\infty}\left(\exists k \in \mathbb{N}.\ \phi_{\pi}^{x_0}(k) \in \mathcal{X}_r \wedge \forall l \in [0,k].\ \phi_{\pi}^{x_0}(l) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) = v(x_0)$,
where $\hat{f}\,(x,d) = f(x,d) \cdot 1_{\mathcal{X}}(x) + x \cdot 1_{\widehat{\mathcal{X}} \setminus \mathcal{X}}(x) + x \cdot 1_{\mathcal{X}_r}(x)$

$\widehat{\mathcal{X}}$ is a set containing the union of the set $\mathcal{X}$ and all reachable states starting from $\mathcal{X}$ within one step

$$\{x \mid x = f(x_0,d), x_0 \in \mathcal{X}, d \in \mathcal{D}\} \cup \mathcal{X} \subseteq \widehat{\mathcal{X}}$$



B. Xue, R. Li, N. Zhan, and M. Fränzle. Reach-avoid analysis for stochastic discrete-time systems. In 2021 American Control Conference (ACC), pages 4879–4885. IEEE, 2021

Equation Relaxation Based Method

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ and $w(x): \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ such that

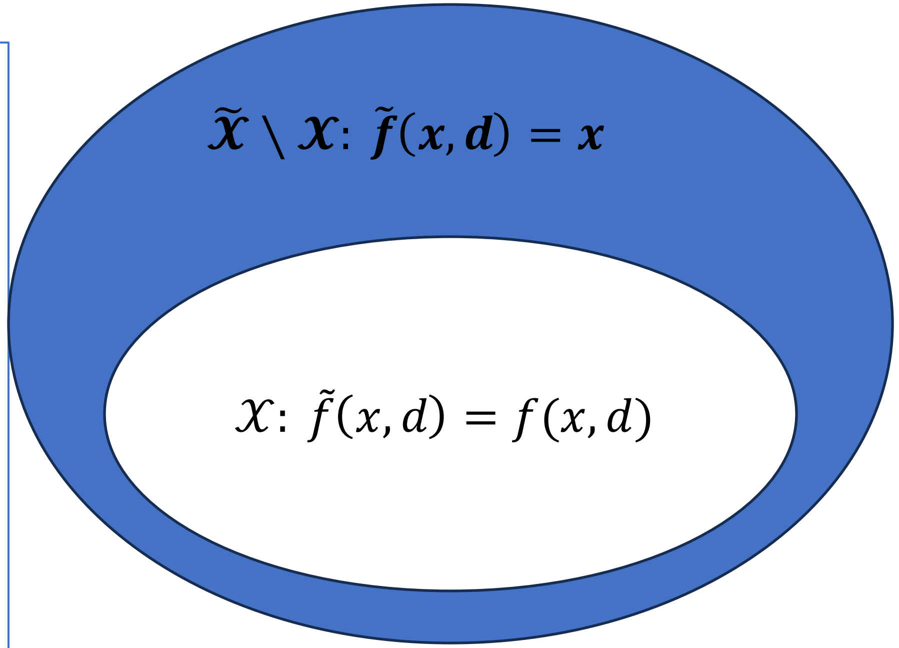
$$\begin{cases} v(x) = \mathbb{E}\left[v\left(\tilde{f}(x,d)\right)\right], & \forall x \in \widetilde{\boldsymbol{\mathcal{X}}}, \\ v(x) = 1_{\widetilde{\mathcal{X}} \setminus \mathcal{X}}(x) + \mathbb{E}\left[w\left(\tilde{f}(x,d)\right)\right] - w(x), & \forall x \in \widetilde{\boldsymbol{\mathcal{X}}}. \end{cases}$$

Then,

$$\mathbb{P}^{\infty}\left(\exists k \in \mathbb{N}.\ \phi_{\pi}^{x_0}(k) \in \widetilde{\mathcal{X}} \setminus \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) = v(x_0).$$

Thus,

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \phi_{\pi}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) = 1 - v(x_0).$$



Equation Relaxation Based Method

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x)\colon \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ and $w(x)\colon \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ such that

$$\begin{cases} v(x) \le 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \ge \mathbb{E}\left[v\left(\tilde{f}(x,d)\right)\right], & \forall x \in \widetilde{\boldsymbol{\mathcal{X}}}, \\ v(x) \ge 1_{\widetilde{\mathcal{X}} \setminus \mathcal{X}}(x) + \mathbb{E}\left[w\left(\tilde{f}(x,d)\right)\right] - w(x), & \forall x \in \widetilde{\boldsymbol{\mathcal{X}}}. \end{cases}$$

Then,

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0 \right) \ge \epsilon_1.$$

Equation Relaxation Based Method

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Then,

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \geq \epsilon_1.$$

$$\tilde{f}(x,d) = f(x,d)\cdot 1_{\mathcal{X}}(x) + x \cdot 1_{\widetilde{\mathcal{X}}\setminus\mathcal{X}}(x)$$

Theorem 3 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x)\colon \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ and $w(x)\colon \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ such that

$$\begin{cases} v(x) \leq 1-\epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \geq \mathbb{E}[v(f(x,d))], & \forall x \in \mathcal{X}, \\ v(x) \geq \mathbb{E}[w(f(x,d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \widetilde{\boldsymbol{\mathcal{X}}} \setminus \mathcal{X}. \end{cases}$$

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Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \widetilde{\mathcal{X}} \to \mathbb{R}$ and $w(x): \widetilde{\mathcal{X}} \to \mathbb{R}$ such that

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Then,

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\pi}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \geq \epsilon_1.$$

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \widetilde{\mathcal{X}} \to \mathbb{R}$ and $w(x): \widetilde{\mathcal{X}} \to \mathbb{R}$ such that

$$\begin{cases} v(x) \geq 1 - \epsilon_2, & \forall x \in \mathcal{X}_0, \\ v(x) \leq \mathbb{E}\left[v\left(\tilde{f}(x,d)\right)\right], & \forall x \in \widetilde{\mathcal{X}}, \\ v(x) \leq 1_{\widetilde{\mathcal{X}}\setminus\mathcal{X}}(x) + \mathbb{E}\left[w\left(\tilde{f}(x,d)\right)\right] - w(x), & \forall x \in \widetilde{\mathcal{X}}. \end{cases}$$

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$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\pi}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \geq \epsilon_1.$$

Theorem 4 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \widetilde{\mathcal{X}} \to \mathbb{R}$ and $w(x): \widetilde{\mathcal{X}} \to \mathbb{R}$ such that

$$\begin{cases} v(x) \geq 1 - \epsilon_2, & \forall x \in \mathcal{X}_0, \\ v(x) \leq \mathbb{E}[v(f(x,d))], & \forall x \in \mathcal{X}, \\ v(x) \leq \mathbb{E}[w(f(x,d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \leq 1, & \forall x \in \widetilde{\mathcal{X}} \setminus \mathcal{X}. \end{cases}$$

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Comparison

Doob's Supermartingale Inequality Based Method

Theorem 1 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exists $v(x)\colon \widetilde{\mathcal{X}} \to \mathbb{R}$ such that

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then

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\right) \geq \epsilon_1.$$

Equation Relaxation Based Method

Theorem 3 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x)\colon \widetilde{\mathcal{X}} \to \mathbb{R}$ and $w(x)\colon \widetilde{\mathcal{X}} \to \mathbb{R}$ such that

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T. Wu, Y. Yu, B. Xia, J. Wang and B. Xue. A Framework for Safe Probabilistic Invariance Verification of Stochastic Dynamical Systems. Arxiv, 2024.

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Sufficient and Necessary Barrier-like Conditions

B. Xue. Sufficient and Necessary Barrier-like Conditions for Safety and Reach-avoid Verification of Stochastic Discrete-time Systems . Arxiv, 2024.

Comparison

Doob's Supermartingale Inequality Based Method

Theorem 1 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exists $v(x): \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ such that

$$\begin{cases} v(x) \leq 1-\epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \geq 0, & \forall x \in \widetilde{\boldsymbol{\mathcal{X}}}, \\ \mathbb{E}[v(f(x,d))] \leq v(x), & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \widetilde{\boldsymbol{\mathcal{X}}} \setminus \mathcal{X}, \end{cases}$$

then

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x_0}}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\right) \geq \epsilon_1.$$

Theorem 2 Let $\mathcal{X}$ be a closed set. If there exists $v(x): \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ such that

$$\begin{cases} v(x) \leq \epsilon_2, & \forall x \in \mathcal{X}_0, \\ \mathbb{E}[v(f(x,d))] - v(x) \leq -\delta, & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \boldsymbol{\partial}\widetilde{\boldsymbol{\mathcal{X}}} \setminus \boldsymbol{\partial}(\widetilde{\boldsymbol{\mathcal{X}}} \setminus \mathcal{X}), \\ v(x) \geq 0, & \forall x \in \widetilde{\boldsymbol{\mathcal{X}}}, \end{cases}$$

then

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x_0}}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \leq \epsilon_2.$$

Equation Relaxation Based Method

Theorem 3 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ and $w(x): \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ such that

$$\begin{cases} v(x) \leq 1-\epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \geq \mathbb{E}[v(f(x,d))], & \forall x \in \mathcal{X}, \\ v(x) \geq \mathbb{E}[w(f(x,d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \widetilde{\boldsymbol{\mathcal{X}}} \setminus \mathcal{X}, \end{cases}$$

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Theorem 4 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ and $w(x): \widetilde{\boldsymbol{\mathcal{X}}} \to \mathbb{R}$ such that

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then

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\right) \geq \epsilon_1.$$

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then

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \leq \epsilon_2.$$

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$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \geq \epsilon_1.$$

Theorem 4 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \widetilde{\mathcal{X}} \to \mathbb{R}$ and $w(x): \widetilde{\mathcal{X}} \to \mathbb{R}$ such that

$$\begin{cases} v(x) \geq 1-\epsilon_2, & \forall x \in \mathcal{X}_0, \\ v(x) \leq \mathbb{E}[v(f(x,d))], & \forall x \in \mathcal{X}, \\ v(x) \leq \mathbb{E}[w(f(x,d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \leq 1, & \forall x \in \widetilde{\mathcal{X}} \setminus \mathcal{X}, \end{cases}$$

then $\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \boldsymbol{\phi}_{\boldsymbol{\pi}}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \leq \epsilon_2.$

$$1 - v(x) \text{ with } w(x) = M\big(1 - v(x)\big), \text{ where } M\delta \geq \sup_{x \in \widetilde{\mathcal{X}}} 1 - v(x)$$

Optimization

Doob's Supermartingale Inequality Based Method

Op1 $\quad \mathrm{Max}_v\ \epsilon_1$

$$\text{s.t.} \begin{cases} v(x) \le 1-\epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \ge 0, & \forall x \in \widetilde{\mathcal{X}}, \\ \mathbb{E}[v(f(x,d))] \le v(x), & \forall x \in \mathcal{X}, \\ v(x) \ge 1, & \forall x \in \widetilde{\mathcal{X}} \setminus \mathcal{X}. \end{cases}$$

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \phi_{\pi}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \ge \epsilon_1.$$

Op2 $\quad \mathrm{Min}_v\ \epsilon_2$

$$\text{s.t.} \begin{cases} v(x) \le \epsilon_2, & \forall x \in \mathcal{X}_0, \\ \mathbb{E}[v(f(x,d))] - v(x) \le -\delta, & \forall x \in \mathcal{X}, \\ v(x) \ge 1, & \forall x \in \partial\widetilde{\mathcal{X}} \setminus \partial(\widetilde{\mathcal{X}} \setminus \mathcal{X}), \\ v(x) \ge 0, & \forall x \in \widetilde{\mathcal{X}}. \end{cases}$$

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \phi_{\pi}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \le \epsilon_2.$$

Equation Relaxation Based Method

Op3 $\quad \mathrm{Max}_{v,w}\ \epsilon_1$

$$\text{s.t.} \begin{cases} v(x) \le 1-\epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \ge \mathbb{E}[v(f(x,d))], & \forall x \in \mathcal{X}, \\ v(x) \ge \mathbb{E}[w(f(x,d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \ge 1, & \forall x \in \widetilde{\mathcal{X}} \setminus \mathcal{X}. \end{cases}$$

$$\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}.\ \phi_{\pi}^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0\right) \ge \epsilon_1.$$

Op4 $\quad \mathrm{Min}_{v,w}\ \epsilon_2$

$$\text{s.t.} \begin{cases} v(x) \ge 1-\epsilon_2, & \forall x \in \mathcal{X}_0, \\ v(x) \le \mathbb{E}[v(f(x,d))], & \forall x \in \mathcal{X}, \\ v(x) \le \mathbb{E}[w(f(x,d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \le 1, & \forall x \in \widetilde{\mathcal{X}} \setminus \mathcal{X}. \end{cases}$$

$$\mathbb{P}^{\infty}\left(\forall \mathrm{k} \in \mathbb{N}.\ \phi_{\pi}^{\mathrm{x}_0}(\mathrm{k}) \in \mathcal{X} \mid \mathrm{x}_0 \in \mathcal{X}_0\right) \le \epsilon_2.$$

They were relaxed into semi-definite programming problems

Example

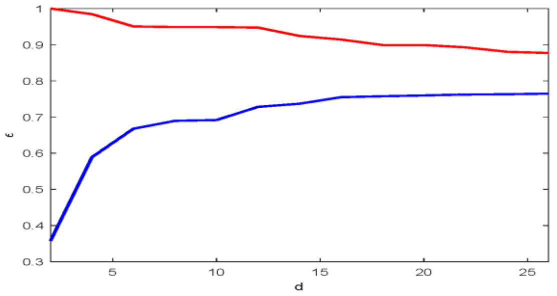
Consider

$$x(l+1) = \big(-0.5 + d(l)\big)x(l)$$

- $d(\cdot): \mathbb{N} \to \mathcal{D} = [-1,1]$ （**uniform distribution**）
- $\mathcal{X} = \{x \mid x^2 - 1 \leq 0\}$
- $\mathcal{X}_0 = \{x \mid (x+0.8)^2 = 0\}$ (i.e., $x_0 = -0.8$)
- $\widehat{\mathcal{X}} = \{x \mid x^2 - 2 \leq 0\}$

The lower and upper bounds of the safety probability obtained by Monte Carlo are $\epsilon_1 = \epsilon_2 = 0.8312$

Op3 and Op4													
d	2	4	6	8	10	12	14	16	18	20	22	24	26
ϵ_1	0.3574	0.5890	0.6678	0.6895	0.6917	0.7281	0.7368	0.7549	0.7575	0.7597	0.7622	0.7630	0.7647
ϵ_2	1.0000	0.9844	0.9505	0.9489	0.9488	0.9474	0.9242	0.9143	0.8991	0.8991	0.8927	0.8804	0.8771
Op1													
ϵ_1	0.3574	0.5890	0.6678	0.6895	0.6917	0.7281	0.7368	0.7549	0.7575	0.7597	0.7622	0.7630	0.7647



Conclusion

Two sets of optimizations for computing lower and upper bounds of the safety probability are introduced:

1. The first one is based on Doob's supermartingale inequality.
2. The second one is based on relaxing an equation that characterizes the exact safety probability.

Thanks!